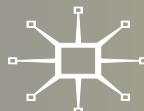


DASAR TRANSFORMASI LAPLACE

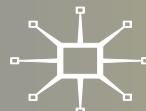
Matematika Industri II



Matematika Industri II

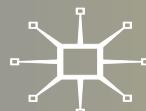
Bahasan

- Transformasi Laplace
- Transformasi Laplace Invers
- Tabel Transformasi Laplace
- Transformasi Laplace dari Suatu Turunan
- Dua Sifat Transformasi Laplace
- Membuat Transformasi Baru
- Transformasi Laplace dari Turunan yang Lebih Tinggi



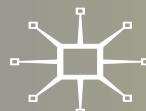
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Transformasi Laplace

- A differential equation involving an unknown function $f(t)$ and its derivatives is said to be *initial-valued* if the values $f(t)$ and its derivatives are given for $t = 0$. These initial values are used to evaluate the integration constants that appear in the solution to the differential equation.
- The Laplace transform is employed to solve certain initial-valued differential equations. The method uses algebra rather than the calculus and incorporates the values of the integration constants from the beginning.



Introduction to Laplace transforms

The Laplace transform

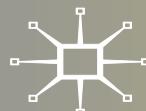
Given a function $f(t)$ defined for values of the variable $t > 0$ then the Laplace transform of $f(t)$, denoted by:

$$L\{f(t)\}$$

is defined to be:

$$L\{f(t)\} = \int_{t=0}^{\infty} e^{-st} f(t) dt$$

Where s is a variable whose values are chosen so as to ensure that the semi-infinite integral converges.

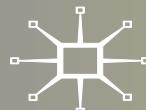


Introduction to Laplace transforms

The Laplace transform

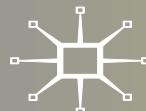
For example the Laplace transform of $f(t) = 2$ for $t \geq 0$ is:

$$\begin{aligned} L\{f(t)\} &= \int_{t=0}^{\infty} e^{-st} f(t) dt \\ &= \int_{t=0}^{\infty} e^{-st} 2 dt \\ &= 2 \left[\frac{e^{-st}}{-s} \right]_{t=0}^{\infty} \\ &= 2(0 - (-1/s)) \\ &= \frac{2}{s} \quad \text{provided } s > 0 \end{aligned}$$



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Introduction to Laplace transforms

The inverse Laplace transform

The Laplace transform is an expression involving variable s and can be denoted as such by $F(s)$. That is:

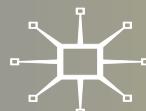
$$F(s) = L\{f(t)\}$$

It is said that $f(t)$ and $F(s)$ form a *transform pair*.

This means that if $F(s)$ is the *Laplace transform* of $f(t)$ then $f(t)$ is the *inverse Laplace transform* of $F(s)$.

That is:

$$f(t) = L^{-1}\{F(s)\}$$



Introduction to Laplace transforms

The inverse Laplace transform

For example, if $f(t) = 4$ then:

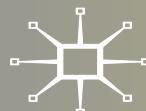
$$F(s) = \frac{4}{s}$$

So, if:

$$F(s) = \frac{4}{s}$$

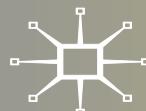
Then the inverse Laplace transform of $F(s)$ is:

$$L^{-1}\{F(s)\} = f(t) = 4$$



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Introduction to Laplace transforms

Table of Laplace transforms

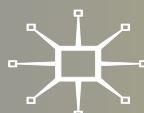
To assist in the process of finding Laplace transforms and their inverses a table is used. For example:

$$f(t) = L^{-1}\{F(s)\} \quad F(s) = L\{f(t)\}$$

$$k \quad \frac{k}{s} \quad s > 0$$

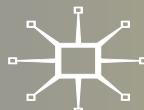
$$e^{-kt} \quad \frac{1}{s+k} \quad s > -k$$

$$te^{-kt} \quad \frac{1}{(s+k)^2} \quad s > -k$$



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Introduction to Laplace transforms

Laplace transform of a derivative

Given some expression $f(t)$ and its Laplace transform $F(s)$ where:

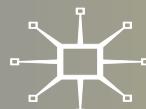
$$F(s) = L\{f(t)\} = \int_{t=0}^{\infty} e^{-st} f(t) dt$$

then:

$$\begin{aligned} L\{f'(t)\} &= \int_{t=0}^{\infty} e^{-st} f'(t) dt \\ &= \left[e^{-st} f(t) \right]_{t=0}^{\infty} + s \int_{t=0}^{\infty} e^{-st} f(t) dt \\ &= (0 - f(0)) + sF(s) \end{aligned}$$

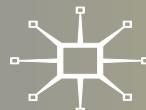
That is:

$$L\{f'(t)\} = sF(s) - f(0)$$



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Introduction to Laplace transforms

Two properties of Laplace transforms

The Laplace transform and its inverse are linear transforms. That is:

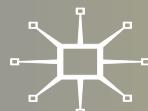
- (1) The transform of a sum (or difference) of expressions is the sum (or difference) of the transforms. That is:

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}$$

$$L^{-1}\{F(s) \pm G(s)\} = L^{-1}\{F(s)\} \pm L^{-1}\{G(s)\}$$

- (2) The transform of an expression that is multiplied by a constant is the constant multiplied by the transform. That is:

$$L\{kf(t)\} = kL\{f(t)\} \quad \text{and} \quad L^{-1}\{kF(s)\} = kL^{-1}\{F(s)\}$$



Introduction to Laplace transforms

Two properties of Laplace transforms

For example, to solve the differential equation:

$$f'(t) + f(t) = 1 \text{ where } f(0) = 0$$

take the Laplace transform of both sides of the differential equation to yield:

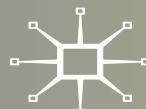
$$L\{f'(t) + f(t)\} = L\{1\} \text{ so that } L\{f'(t)\} + L\{f(t)\} = L\{1\}$$

That is:

$$[sF(s) - f(0)] + F(s) = \frac{1}{s} \text{ which means that } (s+1)F(s) = \frac{1}{s}$$

Resulting in:

$$F(s) = \frac{1}{s(s+1)}$$



Introduction to Laplace transforms

Two properties of Laplace transforms

Given that:

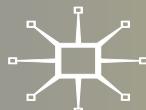
$$F(s) = \frac{1}{s(s+1)}$$

The right-hand side can be separated into its partial fractions to give:

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

From the table of transforms it is then seen that:

$$\begin{aligned} f(t) &= L^{-1} \left\{ \frac{1}{s} - \frac{1}{s+1} \right\} \\ &= L^{-1} \left\{ \frac{1}{s} \right\} - L^{-1} \left\{ \frac{1}{s+1} \right\} \\ &= 1 - e^{-t} \end{aligned}$$



Introduction to Laplace transforms

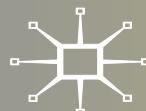
Two properties of Laplace transforms

Thus, using the Laplace transform and its properties it is found that the solution to the differential equation:

$$f'(t) + f(t) = 1 \text{ where } f(0) = 0$$

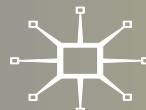
is:

$$f(t) = 1 - e^{-t}$$



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Introduction to Laplace transforms

Generating new transforms

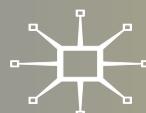
Deriving the Laplace transform of $f(t)$ often requires integration by parts. However, this process can sometimes be avoided if the transform of the derivative is known:

For example, if $f(t) = t$ then $f'(t) = 1$ and $f(0) = 0$ so that, since:

$$L\{f'(t)\} = sL\{f(t)\} - f(0) \text{ then } L\{1\} = sL\{t\} - 0$$

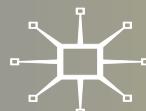
That is:

$$\frac{1}{s} = sL\{t\} \text{ therefore } L\{t\} = \frac{1}{s^2}$$



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Introduction to Laplace transforms

Laplace transforms of higher derivatives

It has already been established that if:

then: $F(s) = L\{f(t)\}$ and $G(s) = L\{g(t)\}$

$$L\{f'(t)\} = sF(s) - f(0) \text{ dan } L\{g'(t)\} = sG(s) - g(0)$$

Now let

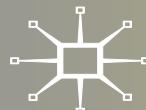
$$g(t) = f'(t) \text{ so } g(0) = f'(0) \text{ and } G(s) = sF(s) - f(0)$$

so that:

$$\begin{aligned} L\{g'(t)\} &= L\{f''(t)\} = sG(s) - g(0) \\ &= s[sF(s) - f(0)] - f'(0) \end{aligned}$$

Therefore:

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$



Introduction to Laplace transforms

Laplace transforms of higher derivatives

For example, if:

then:

$$F(s) = L\{f(t)\}$$

$$L\{f'(t)\} = sF(s) - f(0)$$

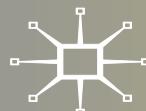
and

$$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

Similarly:

$$L\{f'''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

And so the pattern continues.



Introduction to Laplace transforms

Laplace transforms of higher derivatives

Therefore if:

$$f(t) = \sin kt \text{ so that } f'(t) = k \cos kt \text{ and } f''(t) = -k^2 \sin kt$$

Then substituting in:

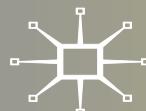
$$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

yields

$$L\{-k^2 \sin kt\} = -k^2 L\{\sin kt\} = s^2 L\{\sin kt\} - s.0 - k$$

So:

$$L\{\sin kt\} = \frac{k}{s^2 + k^2}$$

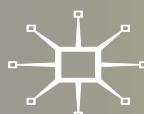


Introduction to Laplace transforms

Table of Laplace transforms

In this way the table of Laplace transforms grows:

$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$	$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$
k	$\frac{k}{s} \quad s > 0$	$\sin kt$	$\frac{k}{s^2 + k^2} \quad s^2 + k^2 > 0$
e^{-kt}	$\frac{1}{s+k} \quad s > -k$	$\cos kt$	$\frac{s}{s^2 + k^2} \quad s^2 + k^2 > 0$
te^{-kt}	$\frac{1}{(s+k)^2} \quad s > -k$		



$$f(t)$$

$$F(s)=\int_0^\infty f(t)e^{-st}\;dt$$

$$f+g$$

$$\alpha F$$

$$\frac{df}{dt}$$

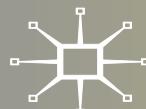
$$sF(s)-f(0)$$

$$\frac{d^k f}{dt^k}$$

$$s^kF(s)-s^{k-1}f(0)-s^{k-2}\frac{df}{dt}(0)-\cdots-\frac{d^{k-1}f}{dt^{k-1}}(0)$$

$$g(t) = \int_0^t f(\tau)\;d\tau$$

$$G(s)=\frac{F(s)}{s}$$



$$g(t)=\int_0^t f(\tau)\;d\tau$$

$$G(s) = \frac{F(s)}{s}$$

$$f(\alpha t),\,\alpha>0$$

$$\frac{1}{\alpha} F(s/\alpha)$$

$$e^{at}f(t)$$

$$F(s-a)$$

$$tf(t)$$

$$-\frac{dF}{ds}$$

$$t^kf(t)$$

$$(-1)^k\frac{d^kF(s)}{ds^k}$$

$$\frac{f(t)}{t}$$

$$\int_s^\infty F(s)\;ds$$

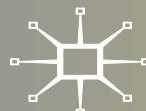
$$g(t)=\left\{\begin{array}{ll} 0 & 0\leq t< T \\ f(t-T) & t\geq T \end{array}\right.,\,T\geq 0\quad G(s)=e^{-sT}F(s)$$



Introduction to Laplace transforms

Learning outcomes

- ✓ Derive the Laplace transform of an expression by using the integral definition
- ✓ Obtain inverse Laplace transforms with the help of a table of Laplace transforms
- ✓ Derive the Laplace transform of the derivative of an expression
- ✓ Solve linear, first-order, constant coefficient, inhomogeneous differential equations using the Laplace transform
- ✓ Derive further Laplace transforms from known transforms
- ✓ Use the Laplace transform to obtain the solution to linear, constant-coefficient, inhomogeneous differential equations of second and higher order.



Reference

- Stroud, KA & DJ Booth. 2003. *Matematika Teknik*. Erlangga. Jakarta

