DASAR TRANSFORMASI LAPLACE

Matematika Industri II

Bahasan

- Transformasi Laplace
- Transformasi Laplace Invers
- Tabel Transformasi Laplace
- Transformasi Lplace dari Suatu Turunan
- Dua Sifat Transformasi Laplace
- Membuat Transformasi Baru
- Transformasi Laplace dari Turunan yang Lebih Tinggi

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Transformasi Laplace

- A differential equation involving an unknown function $f(t)$ and its derivatives is said to be *initial-valued* if the values $f(t)$ and its derivatives are given for $t = 0$. these initial values are used to evaluate the integration constants that appear in the solution to the differential equation.
- The Laplace transform is employed to solve certain initial-valued differential equations. The method uses algebra rather than the calculus and incorporates the values of the integration constants from the beginning.

The Laplace transform

Given a function $f(t)$ defined for values of the variable $t > 0$ then the Laplace transform of $f(t)$, denoted by:

$$L\{f(t)\}$$

is defined to be:

$$L\{f(t)\} = \int_{t=0}^{\infty} e^{-st} f(t)dt$$

Where s is a variable whose values are chosen so as to ensure that the semi-infinite integral converges.

The Laplace transform

For example the Laplace transform of $f(t) = 2$ for $t \geq 0$ is:

$$
\begin{aligned}
L\{f(t)\} &= \int_{t=0}^{\infty} e^{-st} f(t)dt \\
&= \int_{t=0}^{\infty} e^{-st} 2dt \\
&= 2\left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty} \\
&= 2(0-(-1/s)) \\
&= \frac{2}{s} \quad \text{provided } s > 0
\end{aligned}
$$

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The inverse Laplace transform

The Laplace transform is an expression involving variable *s* and can be denoted as such by *F* (*s*). That is:

$$F(s) = L\{f(t)\}$$

It is said that *f* (*t*) and *F* (*s*) form a *transform pair*.

This means that if *F* (*s*) is the *Laplace transform* of *f* (*t*) then *f* (*t*) is the *inverse Laplace transform* of *F* (*s*).

That is:

$$f(t) = L^{-1}\{F(s)\}$$

The inverse Laplace transform

For example, if $f(t) = 4$ then:

$$F(s) = \frac{4}{s}$$

So, if:

$$F(s) = \frac{4}{s}$$

Then the inverse Laplace transform of $F(s)$ is:

$$L^{-1}\{F(s)\} = f(t) = 4$$

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Table of Laplace transforms

To assist in the process of finding Laplace transforms and their inverses a table is used. For example:

$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$	
k	$\dfrac{k}{s}$	$s > 0$
e^{-kt}	$\dfrac{1}{s+k}$	$s > -k$
te^{-kt}	$\dfrac{1}{(s+k)^2}$	$s > -k$

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Laplace transform of a derivative

Given some expression $f(t)$ and its Laplace transform $F(s)$ where:

$$F(s) = L\{f(t)\} = \int_{t=0}^{\infty} e^{-st} f(t)dt$$

then:

$$L\{f'(t)\} = \int_{t=0}^{\infty} e^{-st} f'(t)dt$$

$$= \left[e^{-st} f(t)\right]_{t=0}^{\infty} + s\int_{t=0}^{\infty} e^{-st} f(t)dt$$

$$= (0 - f(0)) + sF(s)$$

That is:

$$\boxed{L\{f'(t)\} = sF(s) - f(0)}$$

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Two properties of Laplace transforms

The Laplace transform and its inverse are linear transforms. That is:

(1) The transform of a sum (or difference) of expressions is the sum (or difference) of the transforms. That is:

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}$$
$$L^{-1}\{F(s) \pm G(s)\} = L^{-1}\{F(s)\} \pm L^{-1}\{G(s)\}$$

(2) The transform of an expression that is multiplied by a constant is the constant multiplied by the transform. That is:

$$L\{kf(t)\} = kL\{f(t)\} \quad \text{and} \quad L^{-1}\{kF(s)\} = kL^{-1}\{F(s)\}$$

Two properties of Laplace transforms

For example, to solve the differential equation:

$$f'(t) + f(t) = 1 \text{ where } f(0) = 0$$

take the Laplace transform of both sides of the differential equation to yield:

$$L\{f'(t) + f(t)\} = L\{1\} \text{ so that } L\{f'(t)\} + L\{f(t)\} = L\{1\}$$

That is:

$$\left[sF(s) - f(0)\right] + F(s) = \frac{1}{s} \text{ which means that } (s+1)F(s) = \frac{1}{s}$$

Resulting in:

$$\boxed{F(s) = \frac{1}{s(s+1)}}$$

Two properties of Laplace transforms

Given that:

$$F(s) = \frac{1}{s(s+1)}$$

The right-hand side can be separated into its partial fractions to give:

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

From the table of transforms it is then seen that:

$$\begin{aligned} f(t) &= L^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} \\ &= L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\} \\ &= 1 - e^{-t} \end{aligned}$$

Two properties of Laplace transforms

Thus, using the Laplace transform and its properties it is found that the solution to the differential equation:

$$f'(t) + f(t) = 1 \text{ where } f(0) = 0$$

is:

$$f(t) = 1 - e^{-t}$$

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Generating new transforms

Deriving the Laplace transform of $f(t)$ often requires integration by parts. However, this process can sometimes be avoided if the transform of the derivative is known:

For example, if $f(t) = t$ then $f'(t) = 1$ and $f(0) = 0$ so that, since:

$$L\left\{f'(t)\right\} = sL\{f(t)\} - f(0) \text{ then } L\{1\} = sL\{t\} - 0$$

That is:

$$\frac{1}{s} = sL\{t\} \text{ therefore } \boxed{L\{t\} = \frac{1}{s^2}}$$

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Laplace transforms of higher derivatives

It has already been established that if:

$$F(s) = L\{f(t)\} \text{ and } G(s) = L\{g(t)\}$$

then:

$$L\{f'(t)\} = sF(s) - f(0) \; dan \; L\{g'(t)\} = sG(s) - g(0)$$

Now let

$$g(t) = f'(t) \text{ so } g(0) = f'(0) \text{ and } G(s) = sF(s) - f(0)$$

so that:

$$L\{g'(t)\} = L\{f''(t)\} = sG(s) - g(0)$$

$$= s\left[sF(s) - f(0)\right] - f'(0)$$

Therefore:

$$\boxed{L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)}$$

Laplace transforms of higher derivatives

For example, if:

$$F(s) = L\{f(t)\}$$

then:

$$L\{f'(t)\} = sF(s) - f(0)$$

and

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Similarly:

$$L\{f'''(t)\} = s^3F(s) - s^2 f(0) - sf'(0) - f''(0)$$

And so the pattern continues.

Laplace transforms of higher derivatives

Therefore if:

$$f(t) = \sin kt \quad \text{so that} \quad f'(t) = k\cos kt \quad \text{and} \quad f''(t) = -k^2 \sin kt$$

Then substituting in:

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

yields

$$L\{-k^2 \sin kt\} = -k^2 L\{\sin kt\} = s^2 L\{\sin kt\} - s.0 - k$$

So:

$$L\{\sin kt\} = \frac{k}{s^2 + k^2}$$

Table of Laplace transforms

In this way the table of Laplace transforms grows:

$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$		$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$	
k	$\frac{k}{s}$	$s>0$	$\sin kt$	$\frac{k}{s^2+k^2}$	$s^2+k^2>0$
e^{-kt}	$\frac{1}{s+k}$	$s>-k$	$\cos kt$	$\frac{s}{s^2+k^2}$	$s^2+k^2>0$
te^{-kt}	$\frac{1}{(s+k)^2}$	$s>-k$			

$f(t)$	$F(s) = \int_0^\infty f(t)e^{-st}\,dt$
$f + g$	$F + G$
$\alpha f \ (\alpha \in \mathbf{R})$	αF
$\frac{df}{dt}$	$sF(s) - f(0)$
$\frac{d^k f}{dt^k}$	$s^k F(s) - s^{k-1} f(0) - s^{k-2}\frac{df}{dt}(0) - \cdots - \frac{d^{k-1} f}{dt^{k-1}}(0)$
$g(t) = \int_0^t f(\tau)\,d\tau$	$G(s) = \frac{F(s)}{s}$

$g(t) = \int_0^t f(\tau)\, d\tau$	$G(s) = \dfrac{F(s)}{s}$
$f(\alpha t),\ \alpha > 0$	$\dfrac{1}{\alpha} F(s/\alpha)$
$e^{at} f(t)$	$F(s-a)$
$t f(t)$	$-\dfrac{dF}{ds}$
$t^k f(t)$	$(-1)^k \dfrac{d^k F(s)}{ds^k}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(s)\, ds$
$g(t) = \begin{cases} 0 & 0 \le t < T \\ f(t-T) & t \ge T \end{cases},\ T \ge 0$	$G(s) = e^{-sT} F(s)$

Learning outcomes

✓Derive the Laplace transform of an expression by using the integral definition

✓Obtain inverse Laplace transforms with the help of a table of Laplace transforms

✓Derive the Laplace transform of the derivative of an expression

✓Solve linear, first-order, constant coefficient, inhomogeneous differential equations using the Laplace transform

✓Derive further Laplace transforms from known transforms

✓Use the Laplace transform to obtain the solution to linear, constant-coefficient, inhomogeneous differential equations of second and higher order.

Reference

- Stroud, KA & DJ Booth. 2003. *Matematika Teknik*. Erlangga. Jakarta